

ON THE EQUALITY CONDITIONS OF THE BRUNN-MINKOWSKI THEOREM

DANIEL A. KLAIN

ABSTRACT. This article describes a new proof of the *equality* condition for the Brunn-Minkowski inequality. The Brunn-Minkowski Theorem asserts that, for compact convex sets $K, L \subseteq \mathbb{R}^n$, the n -th root of the Euclidean volume V_n is concave with respect to Minkowski combinations; that is, for $\lambda \in [0, 1]$,

$$V_n((1 - \lambda)K + \lambda L)^{1/n} \geq (1 - \lambda)V_n(K)^{1/n} + \lambda V_n(L)^{1/n}.$$

The equality condition asserts that, if K and L both have positive volume, then equality holds for some $\lambda \in (0, 1)$ if and only if K and L are homothetic.

Denote n -dimensional Euclidean space by \mathbb{R}^n . Given compact convex subsets $K, L \subseteq \mathbb{R}^n$ and $a, b \geq 0$, denote

$$aK + bL = \{ax + by \mid x \in K \text{ and } y \in L\}.$$

An expression of this form is called a *Minkowski combination* or *Minkowski sum*. Since K and L are convex sets, the set $aK + bL$ is also convex. Convexity also implies that $aK + bK = (a + b)K$ for all $a, b \geq 0$, although this does not hold for general sets. Two sets K and L are *homothetic* if $K = aL + x$ for some $a > 0$ and some point $x \in \mathbb{R}^n$. The n -dimensional (Euclidean) volume of K will be denoted $V_n(K)$.

The Brunn-Minkowski Theorem asserts that the n -th root of the Euclidean volume V_n is concave with respect to Minkowski combinations; that is, for $\lambda \in [0, 1]$,

$$(1) \quad V_n((1 - \lambda)K + \lambda L)^{1/n} \geq (1 - \lambda)V_n(K)^{1/n} + \lambda V_n(L)^{1/n}.$$

If K and L have non-empty interiors, then equality holds for some $\lambda \in (0, 1)$ if and only if K and L are homothetic. This article describes a new proof of this *equality* condition, using a homothetic projection theorem of Hadwiger.

The Brunn-Minkowski Theorem is the centerpiece of modern convex geometry [1, 6, 14, 15]. This theorem encodes as special cases the classical isoperimetric inequality (relating volume and surface area [14, p. 318]), Urysohn's inequality (relating volume and mean width, and strengthening the isodiametric inequality [14, p.

318]), and families of inequalities relating mean projections [14, p. 333]. The concavity implied by (1) leads to families of second-order discriminant-type inequalities for mixed volumes, such as Minkowski's second mixed volume inequality [14, p. 317] and (after substantial additional labor) the Alexandrov-Fenchel inequality [14, p. 327], a difficult and far reaching result with consequences in geometric analysis, combinatorics, and algebraic geometry [4]. Analytic generalizations of the Brunn-Minkowski theorem include the Prekopa-Leindler inequality [3, 10]. The Brunn-Minkowski theorem also serves as the starting point for analogous developments such as the dual Brunn-Minkowski theory for star-shaped sets [11], the L_p -Brunn-Minkowski theory [12], capacitary Brunn-Minkowski inequalities [2], and Brunn-Minkowski inequalities for integer lattices [7]. Of special interest are the *equality* conditions for (1), which imply, for example, the uniqueness of solutions to the Minkowski problem relating convex bodies to measures on the unit sphere [1, 14]. A recent and comprehensive survey on the Brunn-Minkowski inequality and its variations, applications, extensions, and generalizations, can be found in [5].

There are many ways to prove the Brunn-Minkowski Theorem. For the case of compact convex sets, Kneser and Süss used an induction argument on dimension via slicing [14, p. 310]. This proof first verifies the inequality (1), and then addresses the equality case with a more subtle argument.

Hadwiger and Ohmann gave a more general proof by using the concavity of the geometric mean to prove (1) for case of rectangular boxes, and following with a divide-and-conquer argument that extends (1) to finite unions of boxes. They conclude with an approximation step that verifies (1) for all *measurable* (including non-convex) sets [9] (see also [5] and [16, p. 297]). Hadwiger and Ohmann also show that if equality holds in (1) then K and L must both be compact convex sets with at most a set of measure zero removed. Therefore, the question of equality in (1) is addressed completely (up to measure zero) by the case of compact convex sets.

Another especially intuitive proof of the inequality (1) for compact convex sets uses Steiner symmetrization [5, 16]; however, this method relies on approximation and gives no insight into the equality conditions.

In contrast to earlier methods, the proof of the equality condition for (1) presented in this note uses *orthogonal projection* rather than slicing. Sections 1, 2, and 3 provide some technical background. The new proof is presented in Section 4.

1. BACKGROUND

Let \mathcal{K}_n denote the set of compact *convex* subsets of \mathbb{R}^n . If u is a unit vector in \mathbb{R}^n , denote by K_u the orthogonal projection of a set K onto the subspace u^\perp . More generally, if ξ is a d -dimensional subspace of \mathbb{R}^n , denote by K_ξ the orthogonal projection

of a set K onto the subspace ξ . The boundary of a compact convex set K relative to its affine hull will be denoted by ∂K .

Let $h_K : \mathbb{R}^n \rightarrow \mathbb{R}$ denote the support function of a compact convex set K ; that is,

$$h_K(v) = \max_{x \in K} x \cdot v.$$

The standard separation theorems of convex geometry imply that the support function h_K characterizes the body K ; that is, $h_K = h_L$ if and only if $K = L$. If ξ is a subspace of \mathbb{R}^n then the support function of K_ξ within the subspace ξ is given by the restriction of h_K to ξ . Support functions satisfy the identity $h_{aK+bL} = ah_K + bh_L$. (See, for example, any of [1, 14, 16]).

If u is a unit vector in \mathbb{R}^n , denote by K^u the support set of K in the direction of u ; that is,

$$K^u = \{x \in K \mid x \cdot u = h_K(u)\}.$$

If P is a convex polytope, then P^u is the maximal face of P having u in its outer normal cone.

Given $K, L \in \mathcal{K}_n$ and $\epsilon > 0$, the function $V_n(K + \epsilon L)$ is a polynomial in ϵ , whose coefficients are given by *Steiner's formula* [1, 14, 16]. In particular, the following derivative is well-defined:

$$(2) \quad nV_{n-1,1}(K, L) = \lim_{\epsilon \rightarrow 0} \frac{V_n(K + \epsilon L) - V_n(K)}{\epsilon} = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} V_n(K + \epsilon L)$$

The expression $V_{n-1,1}$ is an example of a *mixed volume* of K and L . Since the volume of any set is invariant under translation, it follows from the definition (2) that, for any point $p \in \mathbb{R}^n$,

$$V_{n-1,1}(K + p, L) = V_{n-1,1}(K, L + p) = V_{n-1,1}(K, L).$$

If P is a polytope, then the mixed volume $V_{n-1,1}(P, K)$ satisfies the classical “base-height” formula

$$(3) \quad V_{n-1,1}(P, K) = \frac{1}{n} \sum_{u \perp \partial P} h_K(u) V_{n-1}(P^u),$$

where this sum is finite, taken over all outer unit normals u to the *facets* on the boundary ∂P . Since compact convex sets can be approximated (in the Hausdorff metric) by convex polytopes, the equation (3) implies that, for $K, L, M \in \mathcal{K}_n$ and $a, b \geq 0$,

- $V_{n-1,1}(K, L) \geq 0$,
- $V_{n-1,1}(K, K) = V_n(K)$,
- $V_{n-1,1}(aK, bL) = a^{n-1}bV_{n-1,1}(K, L)$
- $V_{n-1,1}(K, L + M) = V_{n-1,1}(K, L) + V_{n-1,1}(K, M)$,

where the final identity follows from (3) and the linearity of support functions with respect to Minkowski sums. For $n \geq 3$, the function $V_{n-1,1}$ is *not* typically symmetric in its parameters: usually $V_{n-1,1}(K, L) \neq V_{n-1,1}(L, K)$. In particular, the fourth (linearity) property above does not typically hold for the first parameter of $V_{n-1,1}$.

Important special (and more well-known) cases of mixed volumes result from suitable choices of K or L . For example, if B is the unit ball, centered at the origin, then (3) implies that $nV_{n-1,1}(P, B)$ gives the *surface* area of P . A limiting argument then yields the same fact for $nV_{n-1,1}(K, B)$, where K is any compact convex set. Similar arguments imply that, if \overline{ou} denotes the line segment with endpoints at o and a unit vector u , then

$$(4) \quad nV_{n-1,1}(K, \overline{ou}) = V_{n-1}(K_u),$$

the $(n-1)$ -volume of the corresponding orthogonal projection of K . These and many other properties of convex bodies and mixed volumes are described in each of [1, 14, 16].

One especially intuitive proof of the Brunn-Minkowski inequality (1) for compact convex sets uses Steiner symmetrization. Given a unit vector u , view K as a family of line segments parallel to u . Slide these segments along u so that each is symmetrically balanced around the hyperplane u^\perp . By Cavalieri's principle, the volume of K is unchanged. Call the new set $st_u(K)$. It is not difficult to show that $st_u(K)$ is also convex, and that $st_u(K) + st_u(L) \subseteq st_u(K + L)$. A little more work verifies the following intuitive assertion: if you iterate Steiner symmetrization of K through a suitable sequence of directions, these iterations tend to round out the body K to a Euclidean ball B_K having the same volume as the original set K . Meanwhile, it follows from the aforementioned superadditivity relation that $B_K + B_L \subseteq B_{K+L}$, so that

$$\begin{aligned} V_n(K + L)^{1/n} &= V_n(B_{K+L})^{1/n} \\ &\geq V_n(B_K + B_L)^{1/n} \\ &= V_n(B_K)^{1/n} + V_n(B_L)^{1/n} \\ &= V_n(K)^{1/n} + V_n(L)^{1/n} \end{aligned}$$

Technical details behind Steiner symmetrization and the proof outlined above can be found in [16, pp. 306-314]. Once again, because of the approximation step (taking the limit of a sequence of Steiner symmetrizations), it is not clear from this proof exactly when equality would hold in (1).

This matter is addressed by the following theorem.

Theorem 1.1 (Minkowski). *If K and L are compact convex sets with non-empty interiors, then equality holds in (1) if and only if K and L are homothetic.*

While homothety is evidently sufficient, its necessity is far from obvious.

Simple arguments show that the Brunn-Minkowski inequality (1) is equivalent to *Minkowski mixed volume inequality*:

$$(5) \quad V_{n-1,1}(K, L)^n \geq V_n(K)^{n-1} V_n(L),$$

where equality conditions are the same as for (1).

Note that (5) is trivial if either $V_n(K) = 0$ or $V_n(L) = 0$. Moreover, both sides of (5) are positively homogeneous of degree $n(n-1)$ with respect to scaling the body K and positively homogeneous of degree n with respect to scaling the body L . It follows that, for K and L with non-empty interiors, the inequality (5) is equivalent to the assertion that

$$(6) \quad V_{n-1,1}(K, L)^n \geq 1,$$

whenever $V_n(K) = V_n(L) = 1$.

To prove (6), and the equivalent (5), using (1), suppose that $V_n(K) = V_n(L) = 1$, and let

$$f(t) = V_n((1-t)K + tL) = (1-t)^n V_n\left(K + \frac{t}{1-t}L\right),$$

for $t \in [0, 1)$. By (2) and the chain rule,

$$(7) \quad f'(0) = -nV_n(K) + nV_{n-1,1}(K, L) = -n + nV_{n-1,1}(K, L).$$

Since $f^{1/n}$ is concave by (1), we have $f \geq 1$ on $[0, 1)$, while $f(0) = 1$, so that $f'(0) \geq 0$, and $V_{n-1,1}(K, L) \geq 1$.

To prove (1) using (5) the argument is even simpler. Denote $K_t = (1-t)K + tL$. Then

$$\begin{aligned} V_n(K_t) &= V_{n-1,1}(K_t, K_t) = (1-t)V_{n-1,1}(K_t, K) + tV_{n-1,1}(K_t, L) \\ &\geq (1-t)V_n(K_t)^{\frac{n-1}{n}} V(K)^{\frac{1}{n}} + tV_n(K_t)^{\frac{n-1}{n}} V(L)^{\frac{1}{n}}, \end{aligned}$$

by two applications of (5). The inequality (1) then follows after division by $V_n(K_t)^{\frac{n-1}{n}}$.

For a more complete discussion, see any of [1, 5, 14, 16].

2. MIXED AREA

Denote the special case of 2-dimensional volume V_2 by A for area, and denote $V_{1,1}(K, L)$ by $A(K, L)$, the *mixed area*. Unlike the higher dimensional mixed volumes, the mixed area is symmetric in its parameters: $A(K, L) = A(L, K)$. If Δ is a triangle with outward edge unit normals u_1, u_2, u_3 , then

$$(8) \quad A(K, \Delta) = A(\Delta, K) = \frac{1}{2} \sum_i h_K(u_i) |\Delta^{u_i}|,$$

where $|\Delta^{u_i}|$ denotes the length of the i -th edge of the triangle. This identity leads to the following proposition.

Proposition 2.1. *Let $K, L \in \mathcal{K}_2$, and suppose that*

$$(9) \quad A(K, \Delta) = A(L, \Delta)$$

for every triangle Δ in \mathbb{R}^2 . Then K and L are translates.

Proof. Translate K and L so that both lie in the first quadrant of \mathbb{R}^2 and are supported by the coordinate axes. (In other words, slide them both “into the positive corner.”) If e_1 and e_2 respectively denote the unit vectors along the two coordinate axes, we now have $h_K(-e_1) = h_K(-e_2) = 0$, and similarly for L . Since mixed area is invariant under translation, the identity (9) still holds for every triangle Δ .

If u is a unit vector with positive coordinates, let Δ be a right triangle having outward unit normals $-e_1, -e_2, u$. Since $h_K(-e_i) = h_L(-e_i) = 0$, it follows from (8) and (9) that $h_K(u) = h_L(u)$.

If u is a unit vector in one of the other 3 quadrants, a similar argument is then made (using a triangle with unit normals u , one of the $-e_i$, and a suitable choice from the first quadrant) to show that $h_K(u) = h_L(u)$ once again. It follows that $h_K = h_L$, so that $K = L$ after the initial translations of K and L into the positive corner. \square

3. BODIES WITH HOMOTHETIC PROJECTIONS ARE HOMOTHETIC

In Section 4 we give a proof of the equality case for (1) and (5), using a projection argument that relies in the following elementary theorem of Hadwiger [8].

Theorem 3.1 (The Homothetic Projection Theorem). *Suppose that $K, L \in \mathcal{K}_n$ have non-empty interiors, where $n \geq 3$. If K_u and L_u are homothetic for all unit vectors u , then K and L are homothetic as well.*

For completeness of presentation, here is an elementary proof of Theorem 3.1 due to Rogers [13].

Proof. Let $e_i = (0, \dots, 1, \dots, 0)$ denote the unit vector having unit i th coordinate, for $i = 1, \dots, n$.

Translate and scale K and L so that both sets are supported by the positive coordinate halfspace $(e_n^\perp)^+$, and moreover so that $K_{e_n} = L_{e_n}$. The latter is possible, because we are given that K_{e_n} and L_{e_n} are initially homothetic.

It remains to show that, after these translations and dilations, we have $K = L$.

To show this, let $u \in e_n^\perp$ be a unit vector. Recall from the hypothesis of the theorem that $K_u = aL_u + v$ for some $a > 0$ and some $v \in u^\perp$.

Let $\xi = \text{Span}\{u, e_n\}^\perp = u^\perp \cap e_n^\perp$. Since $\xi \subseteq e_n^\perp$ and $K_{e_n} = L_{e_n}$, it follows that

$$L_\xi = K_\xi = aL_\xi + v_\xi.$$

Since $n \geq 3$, we have $\dim \xi = n - 2 \geq 1$. Let V_{n-2} denote volume in \mathbb{R}^{n-2} . Since translation does not change volume,

$$V_{n-2}(L_\xi) = V_{n-2}(K_\xi) = V_{n-2}(aL_\xi + v_\xi) = a^{n-2}V_{n-2}(L_\xi).$$

Since L has non-empty interior, $V_{n-2}(L_\xi) > 0$. It follows that $a = 1$ and $v_\xi = o$. This implies that $v \in \xi^\perp = \text{Span}\{u, e_n\}$; that is, $v = bu + ce_n$ for some $b, c \in \mathbb{R}$. Moreover $v \cdot u = 0$, since we assumed $v \in u^\perp$ to begin with. It follows that $v = ce_n$ for some $c \in \mathbb{R}$.

The positive coordinate halfspace $(e_n^\perp)^+$ supports both K and L , so that $h_K(-e_n) = h_L(-e_n) = 0$. Since $K_u = L_u + v$ and $-e_n \in u^\perp$, we have

$$0 = h_K(-e_n) = h_{K_u}(-e_n) = h_{L_u}(-e_n) + v \cdot (-e_n) = h_L(-e_n) - c = -c,$$

so that $v = ce_n = 0$, and $K_u = L_u$.

We have shown that $K_u = L_u$ for all $u \in e_n^\perp$. If $v \in \mathbb{S}^{n-1}$ then $v^\perp \cap e_n^\perp \neq \emptyset$, so $v \in u^\perp$ for some $u \in e_n^\perp$. It now follows that

$$h_K(v) = h_{K_u}(v) = h_{L_u}(v) = h_L(v).$$

In other words, $h_K(v) = h_L(v)$ for all $v \in \mathbb{S}^{n-1}$, so that $K = L$. \square

4. CONDITIONS FOR EQUALITY

We now have the tools to verify the equality condition for (1) and (5).

Proof of the Equality Condition. For $\lambda \in [0, 1]$, denote $K_\lambda = (1 - \lambda)K + \lambda L$. Suppose that equality holds in (1) for some $\lambda \in (0, 1)$, where K and L have non-empty interiors. Since equality holds for (1) if and only if equality holds for (5) and (6), the homogeneity of mixed volumes allows us to assume without loss of generality that

$$V_n(K) = V_n(L) = V_n(K_\lambda) = V_{n-1,1}(K_\lambda, L) = 1.$$

The concavity of $V_n^{1/n}$ then implies that $V_n(K_\lambda) = 1$ for **all** $\lambda \in [0, 1]$.

Fix a value of $\lambda \in [0, 1]$, and suppose $M \in \mathcal{K}_n$ such that $V_{n-1,1}(K_\lambda, M) \leq 1$. Since $V_{n-1,1}(\cdot, \cdot)$ is Minkowski linear in its second parameter, we have

$$V_{n-1,1}(K_\lambda, (1 - x)L + xM) = (1 - x)V_{n-1,1}(K_\lambda, L) + xV_{n-1,1}(K_\lambda, M) \leq 1$$

for all $x \in [0, 1]$. Since $V_n(K_\lambda) = 1$, it follows from the inequality (5) that

$$\begin{aligned} f(x) &= V_n((1 - x)L + xM) \\ &= V_n((1 - x)L + xM) V_n(K_\lambda)^{n-1} \\ &\leq V_{n-1,1}(K_\lambda, (1 - x)L + xM)^n \leq 1, \end{aligned}$$

for all $x \in [0, 1]$. Since $f(0) = V_n(L) = 1$, it follows that $f'(0) \leq 0$. On computing $f'(0)$ as in (7), we have $V_{n-1,1}(L, M) \leq 1$.

We have shown that if $V_{n-1,1}(K_\lambda, M) \leq 1$, then $V_{n-1,1}(L, M) \leq 1$. The homogeneity of volume now implies that $V_{n-1,1}(L, M) \leq V_{n-1,1}(K_\lambda, M)$ for all M . But the argument above can be repeated, reversing the roles of K_λ and L . Therefore,

$$V_{n-1,1}(K_\lambda, M) = V_{n-1,1}(L, M)$$

for all M and all $\lambda \in [0, 1]$.

If $n = 2$, set $\lambda = 0$. Proposition 2.1 then implies that K and L are translates. This case is the starting point for induction on dimension n .

If $n \geq 3$, then assume the theorem holds in lower dimension. If u is a unit vector, let M denote the line segment \overline{ou} , so that

$$V_{n-1,1}(K_\lambda, \overline{ou}) = V_{n-1,1}(L, \overline{ou}).$$

It follows from (4) that

$$V_{n-1}((K_\lambda)_u) = V_{n-1}(L_u)$$

for all $\lambda \in [0, 1]$. In other words,

$$V_{n-1}((1 - \lambda)K_u + \lambda L_u) = V_{n-1}(L_u)$$

for all $\lambda \in [0, 1]$. This is the equality case of the Brunn-Minkowski inequality in dimension $n - 1$. Since K and L have interior, so do their projections (relative to $(n - 1)$ -dimensional subspaces). It now follows from the induction hypothesis that K_u and L_u are homothetic. Since K and L have homothetic projections in every direction u , it follows from Theorem 3.1 that K and L are homothetic. \square

REFERENCES

- [1] T. Bonnesen and W. Fenchel, *Theory of Convex Bodies*, BCS Associates, Moscow, Idaho, 1987.
- [2] C. Borell, *Capacitary inequalities of the Brunn-Minkowski type*, Math. Ann. **263** (1983), 179–184.
- [3] H. Brascamp and E. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Functional Analysis **22** (1976), 366–389.
- [4] G. Ewald, *Combinatorial Convexity and Algebraic Geometry*, Springer Verlag, New York, 1996.
- [5] R. J. Gardner, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. **39** (2002), 355–405.
- [6] R. J. Gardner, *Geometric Tomography (2nd Ed.)*, Cambridge University Press, New York, 2006.
- [7] R. J. Gardner and P. Gronchi, *A Brunn-Minkowski inequality for the integer lattice*, Trans. Amer. Math. Soc. **353** (2001), 3995–4024.
- [8] H. Hadwiger, *Seitenrisse konvexer Körper und Homothetie*, Elem. Math. **18** (1963), 97–98.
- [9] H. Hadwiger and D. Ohmann, *Brunn-Minkowskischer Satz und Isoperimetrie*, Math. Z. **66** (1956), 1–8.
- [10] R. Henstock and A. M. Macbeath, *On the measure of sum sets, I. The theorems of Brunn, Minkowski and Lusternik*, Proc. London Math. Soc. **3** (1953), 182–194.
- [11] E. Lutwak, *Intersection bodies and dual mixed volumes*, Adv. Math. **71** (1988), 232–261.
- [12] E. Lutwak, *The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem*, J. Differential Geometry **38** (1993), 131–150.
- [13] C. A. Rogers, *Sections and projections of convex bodies*, Portugal. Math. **24** (1965), 99–103.

- [14] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, New York, 1993.
- [15] A. C. Thompson, *Minkowski Geometry*, Cambridge University Press, New York, 1996.
- [16] R. Webster, *Convexity*, Oxford University Press, New York, 1994.

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF MASSACHUSETTS LOWELL, LOWELL, MA 01854
USA

E-mail address: Daniel.Klain@uml.edu